CS 280, Computer Vision (Prof. Jitendra Malik) Jan 27, 1999
Lecture #3 (Curves, Shapes, and Poses) **DRAFT** Notes by Andras Ferencz

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1 Curvature

Not all surfaces are flat. Thus we need a way to characterize the local shape of the surface of an object. That is, given a point P on the surface, we want to know how the surface normal vector $\vec{n}$ changes as we travel on some tangent curve passing through P.

The Normal Curvature of a surface K_n is the directional derivative of $\vec{n}$. That is, we associate a K_n with each direction along the tangent plane of the surface at P. The extrema of Kn are called **principal curvatures** and are usually denoted K_1 and K_2.

Theorem 3.1. *The two principal curvatures K_1 and K_2 are perpendicular. (no proof)*

Corollary 3.2. *K_n for all direction can be determined from the two principal curvatures. It is given by*

$$K_n = K_1 \cos^2\theta + K_2 \sin^2\theta$$

where θ is the angle of K_n measured from K_1.

1.1 Types of Patches

K1 and K2 may be positive (for a convex curve), zero (for a flat line), or negative (for a concave curve). Depending on the sign of K1 and K2, three types of patches (local surface neighborhoods) are defined:

Elliptical patch. *K_1 and K_2 have the same sign (both are positive or negative). Example: Football $K_1 = 1/r$, $0 < K_2 < 1/r$*

Hyperbolic patch. *K_1 and K_2 have the opposite sign (one is positive and the other is negative). Example: Saddle $K_1 > 0$, $K_2 < 0$. Note that Kn must be zero on some line passing through P.*

Parabolic patch. *Either or both K_1 and K_2 are zero. Example: Cylinder $K_1 = 1/r$, $K_2 = 0$*

Note that a line is concave versus convex (K_n positive or negative) by convention only. Flipping the normal vector to the other side changes the sign without changing the curve. Thus in the above only the change in sign between K_1 and K_2 matters.

1.2 Gaussian Curvature

K_n at a point on a surface can clearly be changed by bending the surface. For example, a flat sheet can be bent into the shape of a cylinder, thus changing K_1. However, a flat sheet can not be bent into any arbitrary shape, such as a sphere, without stretching (distorting) the material. What is the difference between these two transformations?

Definition 3.3. *Given a two-dimensional surface in a three-dimensional space, we define two different distances between two points on this surface.* **Intrinsic distance** *is the distance of the two point along the surface.* **Extrinsic distance** *is the distance of the two points in the three-dimensional space.*

Bending the flat sheet described above in any way changes the extrinsic distance of some points. However, bending without stretching does not change the intrinsic distance of any points.

Definition 3.4. *Gaussian Curvature (GC)*

$$GC = K_1 * K_1$$

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Example 4. Plane and cylinder have $GC = 0$. Sphere and Football have $GC > 0$. Saddle has $GC < 0'$.

Theorem 3.5. *Intrinsic distances on a surface are preserved by a transformation if the Gaussian Curvature is unchanged at all points. Bending preserves Gaussian Curvature.*

2 Shape and Pose

An object's **Shape** is preserved under rotation and translation (it's K_n's and thus its GC are also preserved). **Pose**, on the other hand, is the relationship of the object to a set frame - its location (a.k.a. "attitude"). This can be characterized by a distance (3 parameters) and rotation (another 3 parameters).

Characterizing shape can be divided into a local and a global problem. The local problem is relatively easy, and is based on differential geometry developed in the last century by the likes of Gauss and Euler. However, the global problem is very difficult. Topology can help, though two very different objects can have the same topology (a mug and a donut, for example) and in vision we want to identify different objects from a projection even if they have the same topology.

2.1 Local Pose

So far we have been describing the local pose of a patch surrounding point P by the distance r (from P to the focal point F), and the slant σ (the angle between the line $P - F$ and the normal vector $\vec{n}$). However one degree of freedom is missing.

Definition 3.6. *The tilt τ is the direction in with r changes the most (the gradient of r).*

To visualize this, as an angle, the tilt is the direction of $\vec{n}$ projected back onto the viewing plane.

2.2 Back Projection Map

In vision, we are interested in the appearance of objects as they are projected onto a viewing surface. The size and shape of objects appear to change in the image as they move and rotate in the scene. Given (r, σ, τ), we can calculate the appearance of an object in an image.

Imagine a flat infinite surface on which flat objects are placed. As these objects are moved, there are 2 effects on their appearance to consider:

- Shrinkage: as an object is moved farther (r increases), the object appears smaller.
- Foreshortening: as an object is moved farther, the object appears thinner.

Definition 3.7. *The back projection map $F(p)$ is the inverse of the projection of the point P onto the viewing surface (viewsphere):*

$$F : \Sigma \Rightarrow S$$

where Σ is the viewshere and S is the surface in the world. This can be calculated by:

$$F(\vec{p}) = r(\vec{p})\vec{p}.$$

Unfortunately, we often do not know $r(\vec{p})$, making the above calculation difficult.

Definition 3.8. *F_* is the derivative of F.*

As we will see next time, given the right basis

$$F_* = \begin{bmatrix} r/\cos\sigma & 0 \\ 0 & r \end{bmatrix}$$

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