CS 280, Computer Vision (Prof. Jitendra Malik) Feb 3, 1999
Lecture #5 (Rotation and Translation) **DRAFT** Notes by Yogesh Bhumralkar

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Isometry: preserves distances

$$\|\psi(\bar{a}) - \psi(\bar{b})\| = \|\bar{a} - \bar{b}\| \tag{1}$$

Examples:

1. Translation
2. Rotations and Reflections in 2-D

Orthogonal Transformation: Linear Tranformation

$$\psi(\bar{a}) \cdot \psi(\bar{b}) = \bar{a} \cdot \bar{b} \tag{2}$$

Note that for a matrix A:

$$\psi(\bar{a}) = A\bar{a}$$
$$\psi(\bar{a}) \cdot \psi(\bar{b}) = (A\bar{a})^T (A\bar{b}) = \bar{a}^T \bar{b}$$
$$\bar{a}^T A^T A\bar{b} = \bar{a}^T I \bar{b}$$

This implies that:

$$A^T A = I \tag{3}$$

or

$$A^T = A^{-1} \tag{4}$$

So,

$$A = \begin{pmatrix} | & | & | \\ r_1 & r_2 & r_3 \\ | & | & | \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} - & r_1^T & - \\ - & r_2^T & - \\ - & r_3^T & - \end{pmatrix}$$

Since the colunmns of the matrix A are orthonormal, we have:

$$A^{-1}A = \begin{pmatrix} r_1^T r_1 & r_1^T r_2 & r_1^T r_3 \\ r_2^T r_1 & r_2^T r_2 & r_2^T r_3 \\ r_3^T r_1 & r_3^T r_2 & r_3^T r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The components of the comlumn vectors of the matrix A act like the Kronecker Delta function:

$$r_i^T r_j = \delta_{ij}$$

where δ_{ij} is 1 if $i = j$ and 0 otherwise.

$$AA^T = I$$

$$(det\ A) \times (det\ A^T) = (det\ I) = 1$$

$$(det\ A)^2 = 1$$

We note the following important facts:

- det A = +1 implies **Rotations**

- det A = -1 implies **Reflections**

Example of rotation (ie, det A = +1):

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Example of reflection (ie, det A = -1):

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Note that 2 reflections equal a rotation but the converse is not true!

Orthogonal Transformations preserve norms. That is to say:

$$\|A\bar{v}\| = \|\bar{v}\|$$

$$\|A\bar{v}\|^2 = (A\bar{v})^T(A\bar{v}) = \bar{v}^T A^T A\bar{v} = \bar{v}^T\bar{v} = \|\bar{v}\|^2$$

Note that translations do not preserve norms (ie, the distance with respect to the origin changes). Only the distance between pairs of points is preserved with translations. **Orthogonal Transformations preserve distances**. This means that:

$$\|A\bar{v} - A\bar{w}\| = \|A(\bar{v} - \bar{w})\| = \|\bar{v} - \bar{w}\|$$

Since the norm is preserved (as we saw earlier), this is in fact true.

Theorem: 0.1 *Any isometry can be expressed as the combination of an orthogonal transformation followed by a translation as follows:*

$$\psi(\bar{v}) = A\bar{v} + \bar{d}$$

where A represents the orthogonal matrix and $\bar{d}$ is the translation vector.

A proper isometry (also called rigid body motion) is one that requires that $det A = +1$

Group Structure of Rigid Body motions:

$$\psi_2(\bar{v}) = A_2\bar{v} + \bar{d}_2 \qquad \psi_1(\bar{v}) = A_1\bar{v} + \bar{d}_1$$

$\psi_1 \circ \psi_2$ means ψ_1 composed ψ_2 (ie, apply ψ_2 first and then apply ψ_1)

$$\begin{aligned} \psi_1 \circ \psi_2(\bar{v}) &= A_1(A_2\bar{v} + \bar{d}_2) + \bar{d}_1 & (5) \\ &= A_1A_2\bar{v} + A_1\bar{d}_2 + \bar{d}_1 & (6) \\ &= (A_1A_2)\bar{v} + (A_1\bar{d}_2 + \bar{d}_1) & (7) \\ &= A_3\bar{v} + \bar{d}_3 & (8) \end{aligned}$$

where $A_3 = A_1A_2$ and $\bar{d}_3 = A_1\bar{d}_2 + \bar{d}_3$
Assumption: The product of two orthogonal matrices is orthogonal.
Closure Property:
$\psi_1 \circ \psi_2 = \psi_3$ is also a rigid body motion.

Identity Exists:
$A = I$, $\bar{d} = 0$

Inverse exists for each element:
$A_1A_2 = I$ and $\bar{d}_3 = A_1\bar{d}_2 + \bar{d}_1 = 0$. This means that ψ_1 is the inverse of ψ_2. $A_1 = A_2^T$ and $\bar{d}_2 = -A_1^{-1}\bar{d}_1$

Associativity:
Rotations constitute a group but reflections by themselves don't!

Jargon:
$O(n)$: group of orthogonal transformations from $R^n \longmapsto R^n$
$SO(n)$: special, orthogonal group. $det = +1$ (ie, rotations). No translations.
$SE(n)$: group of proper rigid body motions. Proper orthogonal (ie, $det = +1$) followed by Translation. The E in SE stands for Euclidean.

Rotation Matrices: 3D

- Rotation about z-axis by θ:
$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- Rotation about x-axis by θ:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Vector (Cross) Product: Denoted $\bar{a} \times \bar{b}$ or $\bar{a} \wedge \bar{b}$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a2b_1 \end{pmatrix} \tag{9}$$

Another way of looking at this is to imagine multiplying the vector $\bar{b}$ by a matrix $\hat{a}$ such that:

$$\hat{a} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

We immediately see that that the cross product operation is equivalent to this matrix multiplication (ie, $\bar{a} \times \bar{b} = \hat{a}\bar{b}$). A is called a **Skew-Symmetric Matrix** because $\hat{a}^T = -\hat{a}$

Definition: A skew-symmetric matrix S is one that satisfies the condition $S^T = -S$ (ie, $S_{ij} = -S_{ji}$ and $S_{ii} = 0$)

Given a rotation matrix R, what is the axis/amount of rotation (and vice versa)? Consider Rotation about the axis ω (assume $\|\omega\| = 1$):

$$\dot{q}(t) = \bar{\omega} \wedge \bar{q}(t)$$
$$\bar{q}(t) = e^{\bar{\omega}t} \cdot \bar{q}(0) = R(\bar{\omega}, t) \cdot \bar{q}(0)$$

Recall,

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots \tag{10}$$

Define the following quantities:

- Axis of rotation is $\bar{\omega}$
- Amount of rotation is θ
- The Orthogonal Rotation Matrix is $R = e^{\hat{\omega}\theta}$ where $\hat{\omega}$ is the skew-symmetric matrix.

Rodrigues Formula: in 3-D

$$R(\omega, \theta) = I + \hat{\omega} \sin\theta + \hat{\omega}^2(1 - \cos\theta) \tag{11}$$

where $\hat{\omega}$ is the axis of rotation and θ is the amount of rotation.